Computación y Sistemas Vol. 9 Núm. 3, pp. 227-242

Sampling – Reconstruction Procedure of Gaussian Fields

Procedimiento para el Muestreo y Reconstrucción de Campos Gausianos

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Article received on June 04, 2004; accepted on August 11, 2005

Abstract

The description of the optimal Sampling – Reconstruction Procedure (SRP) of Gaussian fields is given on the basis of the conditional mean rule when the quantity of samples is limited. The Gaussian fields are described by two types of space covariance function: exponential and Gaussian. A lot of both reconstruction and reconstruction error surfaces are obtained by numerical calculation. We changed the type of the covariance functions; the type of sampling (uniform: triangular, square, etc. and non – uniform: polar, spiral, and arbitrary); the quantity of the samples; the distances between the samples; and radii of the covariance functions of both axes. We demonstrate how all above mentioned factors influence on principal optimal SRP characteristics. The results of the calculations have clear interpretations.
Keywords: Gaussian Fields, Uniform and Non - Uniform Sampling, Reconstruction Functions, Reconstruction Error Functions**.**

Resumen

La descripción del Procedimiento óptimo de Muestreo – Reconstrucción de los procesos Gaussianos esta dada en base a la regla de la media condicional cuando la cantidad de las muestras es limitada. Los Campos Gaussianos están descritos por dos diferentes funciones espaciales de covarianza: exponencial y Gaussiana. Varias superficies de reconstrucción y de error de reconstrucción son obtenidas a partir de los cálculos numéricos. Cambiamos el tipo de las funciones de covarianza; el modo de muestreo (uniforme: triangular, cuadrada, etc. y no uniforme: polar, espiral y arbitraria); la cantidad de muestras; la distancia entre las muestras; el radio de las funciones de covarianza en ambos ejes. Demostramos como estos factores influyen en las principales características del Procedimiento óptimo de Muestreo - Reconstrucción.
Palabras claves: Campos Gaussianos, Muestreo Uniforme y no Uniforme, Funciones de Reconstrucción, Funciones de Error de Reconstrucción.
2000 Mathematics subjects classification –60Hxx, 94A20

1 Introduction

The description of the Sampling-Reconstruction Procedure (SRP) of both deterministic images and stochastic fields has been discussed in many publications. Here we note the papers.(Petersen and Middleton, 1965) –(Bourgeois, 2001), (Pogany, 1999) – (Klesov, 1985). In Clark (1985) and Stark (1993), the SRP of some deterministic images is considered. The papers (Petersen, 1965) – (Zeevi, 1993), and (Bourgeois, 2001), are devoted to the investigation of some different aspects of the SRP of random fields. Usually, the random fields are characterized by a covariance function or a power spectral density. *A probability density function (pdf) of fields is not discussed by the authors of the mentioned papers.* The work (Poganu, 1999) is a review of the publications with almost sure restoration algorithms for both the stochastic processes and random fields. The main feature of this approach is related to some generalizations of the so-called Kotel'nikov-Shannon's series and the problems of the convergence of this series in the multidimensional case (see also (Pogany, 1995) – (Klesov, 1985), and some other references in Pogany (1999)). Different kinds of sampling are used in all of these papers: uniform (triangular, square, pentagonal, etc.) and non-uniform (polar, spiral, etc.). In Clark, (1985), Zeevi (1993), and Bourgeois (2001), the main attention is put on the transformation of the non-uniform sample set into a uniform sample set in order to attain simplification in an analysis.

The problem of *the calculation* of *the reconstruction error is very difficult in any case*. For instance, in Petersen, (1965), the calculation of the error reconstruction function is given for a *one-dimensional* case, i. e. for a random process but not for a random field. In Zeevi, (1993), some *bounds* for the reconstruction of *one-dimensional* errors were obtained. The reason for these difficulties is the lack of any information about pdf of the fields under consideration.

The present paper is devoted to the statistical description of the SRP of some *Gaussian* fields on the basis of the *multidimensional* conditional mathematical expectation rule. As is well known (see, for instance, Cramer, (1946)) this rule provides the minimum of the mean square error. This approach was applied in the statistical description of the optimal SRP of some stochastic *processes* and their transformations, when the *sample set is limited*. The case of a limited set of samples is very important both for the practice and for the theory. The present paper is a generalization of the conditional mean rule on the multidimensional case - the Gaussian random fields.

We consider Gaussian random fields with two types of space covariance functions: exponential and Gaussian. The fields can be isotropic or anisotropic. This feature is analytically reflected by equal or different radii of the covariance in the expressions of the covariance functions. The corresponding spectral density functions are *infinite*. The quantity of samples is arbitrary and limited. The location of the samples is arbitrary, i.e., the distance between samples can be arbitrary as well. It is necessary to know the exact coordinates of all samples.

One can emphasize a very important fact. In order to describe the Gaussian fields *completely* it is *sufficient* to use the usual statistical characteristic - a space covariance function and the mathematical expectation. For this reason we do not need any other special statistical characteristics. The difference between the present paper and many other papers is that: i) we apply the conditional mean rule; ii) we analyze the optimal SRP algorithms with an arbitrary number of samples; and iii) we concretize the type of multidimensional pdf as the Gaussian pdf. We give a *detailed optimal* SRP description of Gaussian fields, i.e. we determine the optimal reconstruction surfaces and the minimum reconstruction error surfaces in many very important practical cases. Therefore, the present paper has an applied character.

Non-Gaussian fields must be characterized by a special multidimensional pdf or by many moment or cumulant multidimensional functions of *high* orders. The calculation of both the conditional mean and the conditional variance is very difficult in this case.

Some results of the present paper were orally presented by Kazakov (2003).

2 General expressions

For the sake of methodology we first discuss some principal expressions for stochastic Gaussian *processes* and after that some expressions for Gaussian *fields* will be considered.

2.1 The case of the Gaussian processes

Let us consider the general case of a *non-stationary* Gaussian process $x(t)$ with the mathematical expectation $m(t)$, the variance $\sigma^2(t)$, and the covariance function $K(t_1, t_2)$. This is the complete information about the given process because one can write the exact expression for the multidimensional pdf of the arbitrary order *m*

$$\omega_m\left[x(t_1), x(t_2), ..., x(t_m)\right] = (2\pi)^{-m/2}\left[\det \| K(t_i, t_l) \|\right]^{-1/2} \times \exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{l=1}^{m}\left[x(T_i) - m(T_i))\right] a_{il}\left[x(T_l) - m(T_l))\right]\right\}, \qquad (1)$$

where $\det \| K(t_i, t_l) \|$ is the determinant of the covariance matrix

$$\| K(t_i,t_l)\| = \begin{Vmatrix} K(t_1,t_1) & K(t_1,t_2) & \dots & K(t_1,t_m) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ K(t_m,t_1) & K(t_m,t_2) & \dots & K(t_m,t_m) \end{Vmatrix} \tag{2}$$

and $\|a_{il}\|$ is the inverse of the covariance matrix, i.e.,

$$\|a_{il}\| = \|K(t_i,t_l\|^{-1}, \sum_{l=1}^{m} a_{il} K(t_l,t_k) = \delta_{ik}, \quad (\delta_{ik} = 1 \text{ for } i = k, \delta_{ik} = 0 \text{ for } i \neq k \tag{3}$$

We fix the set of the samples **X, T** $= \{x(T_1), x(T_2), ..., x(T_N)\}$. Then the conditional pdf will be a Gaussian one also. The main statistical characteristics of this conditional process are known (see, for example, the pages 44 – 47 of Stratonovich, (1963)):

$$\tilde{m}(t) = m(t) + \sum_{i=1}^{N}\sum_{l=1}^{N} K(t,T_i) a_{il}\left[x(T_l) - x(T_l)\right] \tag{4}$$

$$\tilde{\sigma}^2(t) = \sigma^2(t) - \sum_{i=1}^{N}\sum_{l=1}^{N} K(t,T_i) a_{il} K(T_l,t) \tag{5}$$

$$\tilde{K}(t_1,t_2) = K(t_1,t_2) - \sum_{i=1}^{N}\sum_{l=1}^{N} K(t,T_i) a_{il} K(T_l,t_2) \tag{6}$$

The conditional Gaussian process $\tilde{x}(t)$ is completely described by the expressions (4) – (6).

It must be emphasized that *all formulas* of this Subsection are valid for a *non-stationary* Gaussian process and for an *arbitrary* set of samples **X, T**. This means: i) we need to know all moment functions $m(t)$, $\sigma^2(t)$, $K(t_1,t_2)$ of a non-stationary process $x(t)$; ii) the set of samples **X, T** can have an *arbitrary number* of samples *N* and any *arbitrary distance* between samples $T_i - T_{i-1}$. We choose the conditional expectation mean $\tilde{m}(t)$ as the *optimal* reconstruction function which provides the *minimum* of the square error $\tilde{\sigma}^2(t)$ of the estimation of a given process $x(t)$ for any time *t* when the set of samples **X, T** is given. One can notice here that the quantity of samples is limited by *N*. Therefore , the reconstruction procedure of a complete realization of the process $x(t)$ $(0 < t < \infty)$ must be carried out by the renewal of the given set of samples **X, T**. There are some different variants of this renewal. We use the variant when the quantity of samples *N* is constant and the renewal of the given set is fulfilled by a shift: one (or some) of the past samples is (are) canceled and one (or some) new sample(s) is (are) entered into the set of samples. From (4) one can notice that the reconstruction function $\tilde{m}(t)$ depends on both all of samples $x(T_l)$ $(l = 1,..., N)$ and the unconditional mathematical expectation $m(t)$, $m(T_l)$ $(l = 1,..., N)$.The error reconstruction function $\tilde{\sigma}^2(t)$ does not depend on the values of samples, but depends on both the coordinates $T_l\,(l = 1,..., N)$ of samples and the unconditional variance $\tilde{\sigma}^2(t)$. These main statistical characteristics of the SRP $\tilde{m}(t)$ and $\tilde{\sigma}^2(t)$ depend on both the covariance moments between *all pairs* of the samples $K(T_i,T_j)$ (this dependence is reflected by the elements a_{ij} of the inverse covariance matrix) and the covariance function $K(t,T_l)$, $K(T_l,t)$ between the random variable $x(t)$ with the current time *t* and *all* samples $x(T_l)$. It is clear that the reconstruction function $\tilde{m}(t)$ is equal to the *values of samples* $x(T_l)$ $(l = 1,..., N)$ when

$t = T_l$. It is also clear that the error reconstruction function $\tilde{\sigma}^2(t)$ is equal to *zero* in the all moments $T_l\,(l = 1,..., N)$ of samples and it tends to the a priori variance when $t - T_N \to \infty$ (this is the extrapolation case).

This approach was used in the statistical description of the SRP of stochastic processes of different types. In particular, the SRP of *non-stationary* Gaussian processes was investigated by Kazakov (1997 and 2002). Some problems of the SRP description of the *stationary* Gaussian processes were discussed in the papers written by Kazakov *et al.* (1994 and 1995). In this case in the formulas (4) and (5) it is necessary to have *m(t) = m*, $\sigma^2(T) = \sigma^2$, and $K(T, t_l) = K(T - t_l)$, $K(T_l, t) = K(T_l - t)$. The SRP of one-dimensional Gaussian processes is considered by Kazakov *et al.* (1994). The SRP of multidimensional Gaussian processes (with some two-dimensional examples) was investigated by Kazakov *at al.* (1995). Papers written by Kazakov (1988 and 2001) were devoted to the SRP of some *non-Gaussian* processes.

2.2 The case of the Gaussian fields

Generally, random fields can be changed both in space and in time. We restrict our analysis to the case when the field does not depend on time. It depends on two coordinates $\xi(x, y)$. We study the case of the Cartesian coordinates *x* and *y*. Hence, the field is represented by an infinity of surfaces as separate realizations. In the Gaussian case the field can be completely determined by its mathematical expectation $\langle \xi(x, y) \rangle = m(x, y)$ and the space covariance function $K(x, x + \Delta x; y, y + \Delta y)$. Below we consider the stationary case when $m(x, y) = m$ and $K(x, x + \Delta x; y, y + \Delta y) = K(\Delta x, \Delta y)$ .It is natural that $K(\Delta x = 0, \Delta y = 0) = \sigma^2$. This is the variance of the field.

We choose two types of covariance functions: Gaussian and exponential. The Gaussian covariance function corresponds to some smooth fields. Its mathematical expression is

$$K(\Delta x, \Delta y) = \sigma^2 \exp\left[-\left(\alpha_x (\Delta x)^2 + \alpha_y (\Delta y)^2\right)\right] \tag{7}$$

where α_x and α_y are the coefficients determined by a slope of change of a covariance function along the axes *x* and *y*.

The broken fields are characterized by the exponential covariance function. Its mathematical expression is

$$K(\Delta x, \Delta y) = \sigma^2 \exp\left[-(\beta_x |\Delta x| + \beta_y |\Delta y|)\right] \tag{8}$$

In order to give the physical interpretation of the coefficients β_x and β_y we fix one coordinate (for instance *y*) and make a section. In this section we have a stochastic Gaussian process $\xi(x)$ with the exponential covariance function $K(\Delta_x) = \sigma^2 \exp[-(\beta_x |\Delta_x|]$.As is known, this covariance function describes the Markov Gaussian process with the frequency band of the power spectrum β_x or with "the covariance time " $1/\beta_x$. It is clear that in our case it is necessary to call this parameter “the covariance radius” ρ_x along the axis *x*.

We remember that the Markov Gaussian process is non-differentiable, hence, its realizations are broken and its spectrum is rather wide. If $\alpha_x = \alpha_y$ and $\beta_x = \beta_y$ then the Gaussian fields are isotropic. In other cases the fields are un-isotropic.

In order to get surfaces of both the optimal reconstruction function and the minimum reconstruction error function we need to enter the *one* two-dimensional random variable with the current coordinates $\xi(x, y)$. This variable is the conditional variable with respect to the set of *N* fixed samples: $\xi(x_1, y_1), \xi(x_2, y_2), ..., \xi(x_N, y_N)$. Then, we change our designations in expressions (4) and (5): the current point of the field is $\xi(x, y)$ and the fixed set of samples is $\Xi = \{\xi(x_1, y_1), \xi(x_2, y_2),..., \xi(x_N, y_N)\}$. Hence, instead of the expressions (4) and (5) we have:

$$\tilde{m}(x, y) = m(x, y) + \sum_{i=1}^{N}\sum_{l=1}^{N} K(x, y; x_i, y_i)\, a_{il} \left[\xi(x_l, y_l) - m(x_l, y_l)\right] \tag{9}$$

$$\tilde{\sigma}^2(x, y) = \sigma^2(x, y) - \sum_{i=1}^{N}\sum_{l=1}^{N} K(x, y; x_i, y_i)\, a_{il} K(x_l, y_l; x, y) \tag{10}$$

As one can see, we use the same indexes for both Cartesian coordinates of each sample.

For the sake of simplification we put the mean of field equal to zero $m(x, y) = \langle \xi(x, y) \rangle = 0$, and the variance equal to one $\sigma^2(x, y) = \sigma^2 = 1$. Then formulas (9) and (10) will be simplified:

$$\tilde{m}(x, y) = \sum_{i=1}^{N}\sum_{l=1}^{N} K(x, y; x_i, y_i) a_{il} \xi(x_l, y_l) \text{ s} \tag{11}$$

$$\tilde{\sigma}^2(x, y) = 1 - \sum_{i=1}^{N}\sum_{l=1}^{N} K(x, y; x_i, y_i) a_{il} K(x_l, y_l; x, y) \text{ s} \tag{12}$$

All the following results of the calculations are based on expressions (11) and (12)

3 The uniform SRP

Now we apply the general expressions for some types of uniform SRP. In this case the points of the field discretization are located at the angles of triangles, of squares, of pentagons, etc. The quantity of samples, their Cartesian coordinates, and their values must be known.

The surfaces of the two *reconstruction functions* are presented in Fig. 1 and Fig. 2 when the quantity of samples is equal to 4 and the points of samples are located at the angles of a square. The surface in Fig. 1 corresponds to the field with the *Gaussian* covariance function (7). The parameters of the calculation are the following: The number of the samples is 5. The coordinates of the samples are: $x_1 = y_1 = -0.5$; $x_2 = -0.5, y_2 = 0.5$; $x_3 = y_3 = 0.5$; $x_4 = 0.5$, $y_4 = -0.5$; $x_5 = y_5 = 0$. The covariance radii are equal: $\rho_x = \rho_y = 1$. The reconstruction function of the field with the *exponential* covariance function (8) is presented in Fig. 2. Both the coordinates and values of the samples are equal in Fig. 1 and Fig. 2. As one can see, the reconstruction surfaces have a difference. The surface in Fig. 1 is smooth; it does not have any fractures. The surface in Fig. 2 is characterized by the presence of many fractures. The reason for these differences is determined by the different properties of both covariance functions (7) and (8). These differences will be observed in all graphs in this paper; therefore, we will not discuss below these features of the graphs.

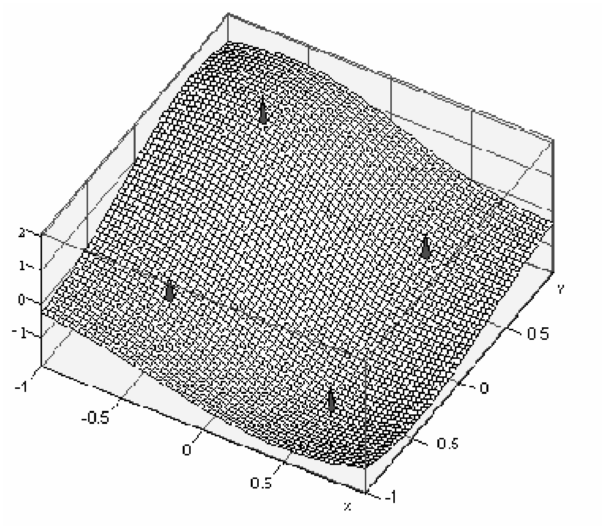

Fig. 1. The reconstruction function for uniform sampling. The covariance function is Gaussian.

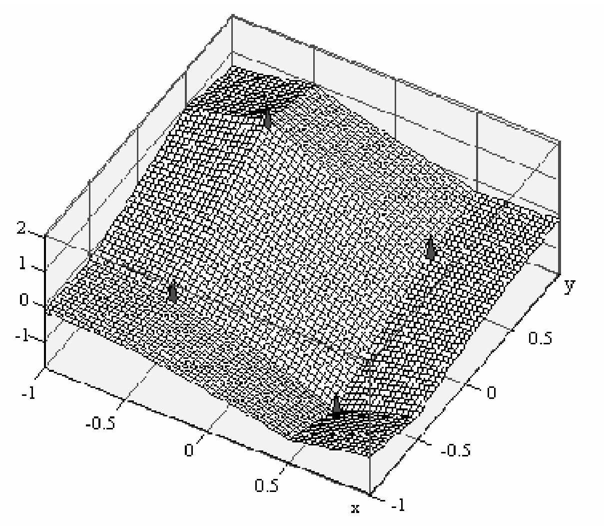

Fig. 2. The reconstruction function for uniform sampling. The covariance function is exponential.

For the square sampling the surfaces of the three *reconstruction error functions* are presented in Fig. 3, 5, 7. Fig. 4, 6, 8 illustrate the sections of these surfaces. The graphs in Fig. 3, 4 and Fig. 5, 6 are related to sampling with the same parameters as in Fig. 1 and 2. As one can see, the reconstruction errors are equal to zero in the points of samples. The errors are increased if the current point is moved off the sample points. This is the reason for a local maximum of the error in the centers of Fig. 3 - 6. It is natural that the errors are increased to one if the coordinates of the current point are moved off the sample points. The graphs in Fig. 7 and 8 are characterized by another case. The parameters of the calculation are the following: $x_1 = y_1 = -0.25$; $x_2 = -0.25, y_2 = 0.25$; $x_3 = y_3 = 0.25$; $x_4 = 0.25, y_4 = -0.5$; $x_5 = y_5 = 0$. The covariance radii are equal: $\rho_x = \rho_y = 1$. Here the distances between the sample points are smaller then the distances in the cases Fig. 1 - 6, but the covariance radii are the same and are equal to 1. Here there is not big maximum in the centers of the Fig. 7 and 8.

It is natural that the statistical dependences between the current point around the center of the square and the sample points are stronger. Hence, the reconstruction error is smaller, but not equal to zero. Therefore, one can not see the maximum in the center of the square in the chosen scale of the graphs.

The *anisotropic* case is illustrated by Fig. 9 and 10. The locations of the samples are the same as in Fig. 1- 6. But the radii of the covariations are different. Here, instead of $\rho_x = \rho_y = 1$ we have $\rho_x = 1.2$ and $\rho_y = 0.8$. The graphs on Fig. 9 and 10 show a very natural result: the reconstruction error along the axis *y* has the smaller values in comparison with the error along the axis *x* if the distances between the current point and the sample points are the same.

The graphs of error reconstruction functions with the triangular sampling when the quantity of samples is equal to 7 are presented in Fig. 11 - 14. In these Figures the parameters of the calculation are as follows: $x_1 = -1.3; y_1 = 0$; $x_2 = -0.65; y_2 = -1.125$; $x_3 = 0.65; y_3 = -1.125$; $x_4 = 1.3; y_4 = 0$; $x_5 = 0.65; y_5 = 1.125$; $x_6 = -0.65; y_6 = 1.125$; $x_7 = y_7 = 0$. The covariance functions are different: in Fig. 11 and 12 we have Gaussian covariance and in Fig. 13 and 14 - exponential. The covariance radii are equal: $\rho_x = \rho_y = 1$. The shapes of all curves are clear, so it is not necessary to give any other explanation.

It is quite possible to consider some other types of uniform sampling.

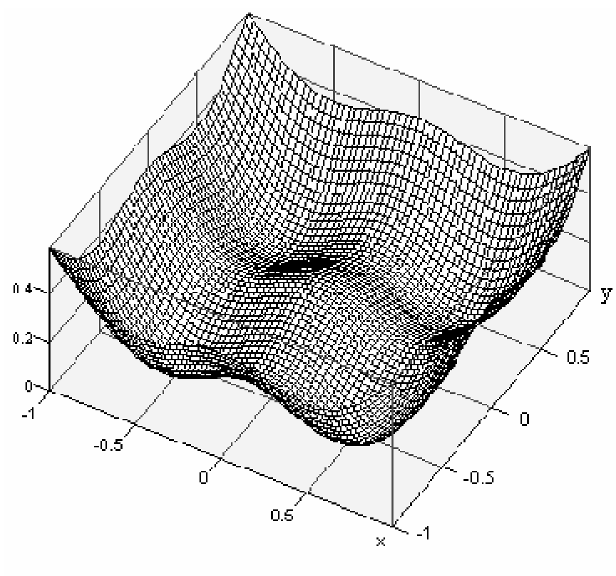

Fig. 3. The error reconstruction function for uniform sampling. The covariance function is Gaussian.s

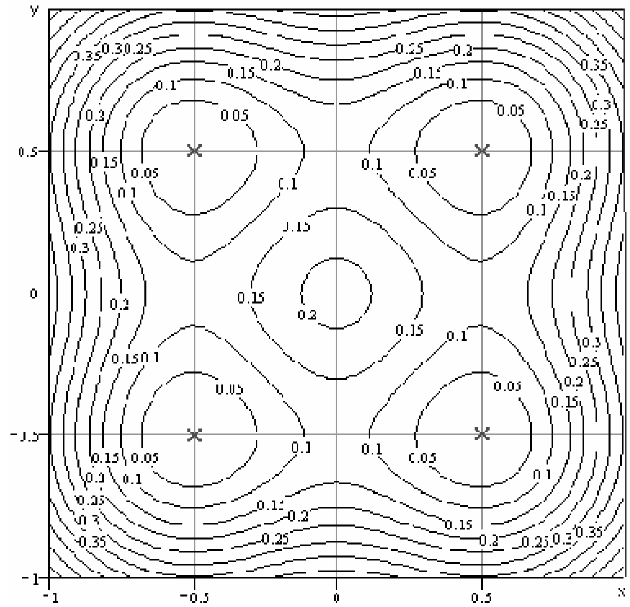

Fig. 4. The sections of error reconstruction function for uniform sampling. The covariance function is Gaussian.s

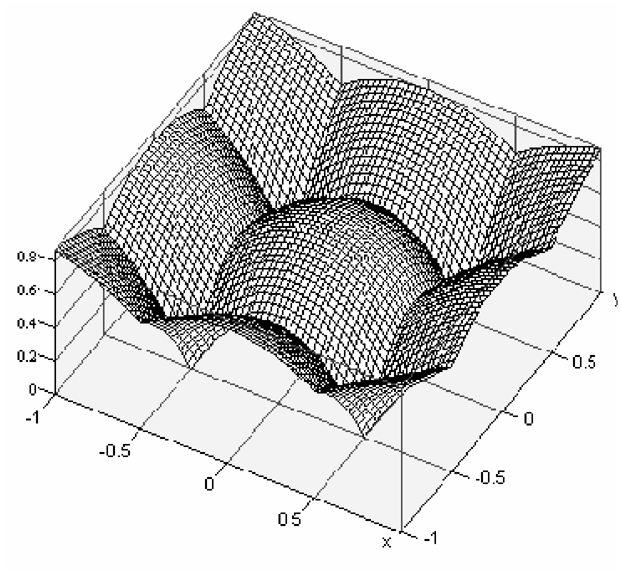

Fig. 5. The error reconstruction function for uniform sampling. The covariance function is exponential.

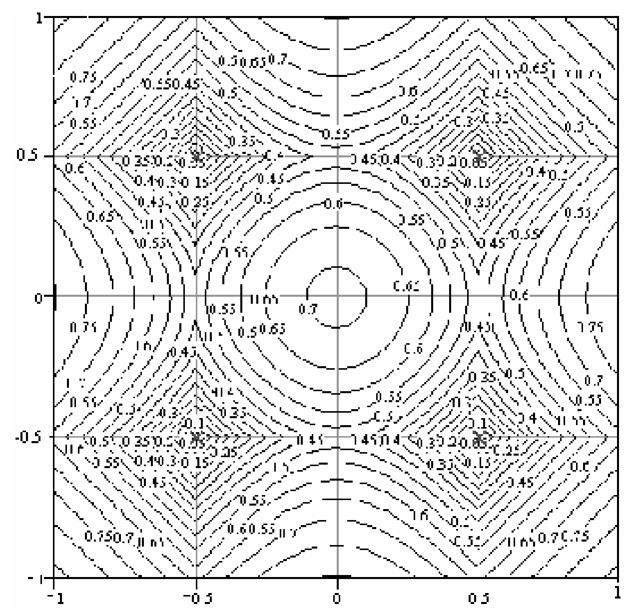

Fig. 6. The sections of error reconstruction function for uniform sampling. The covariance function is exponential.

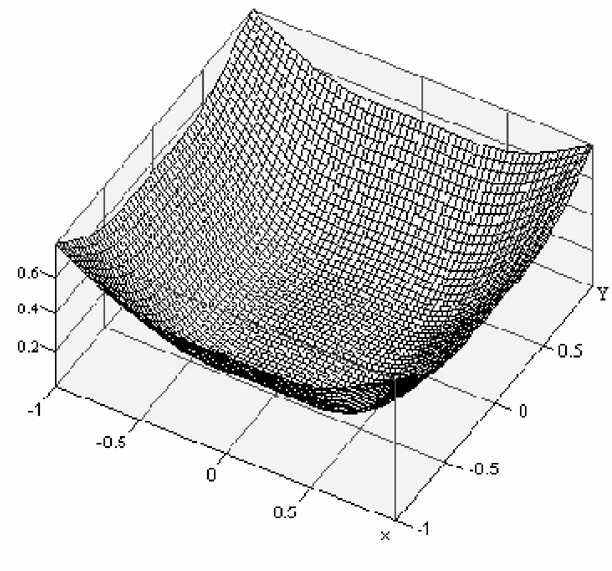

Fig. 7. The error reconstruction function for uniform sampling. The covariance function is Gaussian.

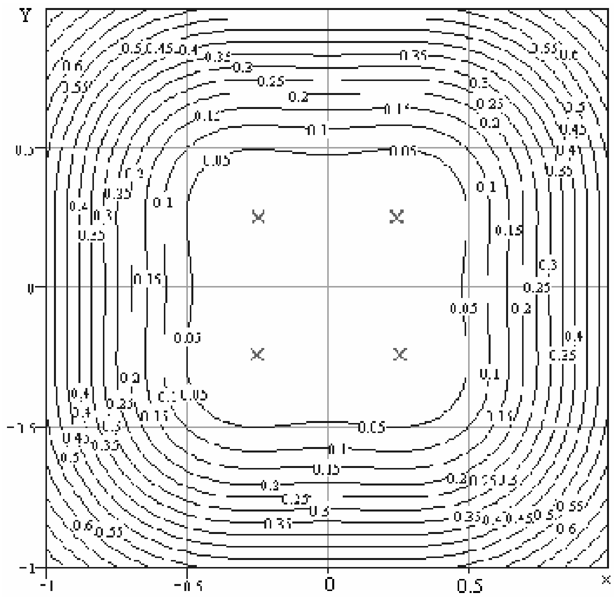

Fig. 8. The sections of error reconstruction function for uniform sampling. The covariance function is Gaussian.

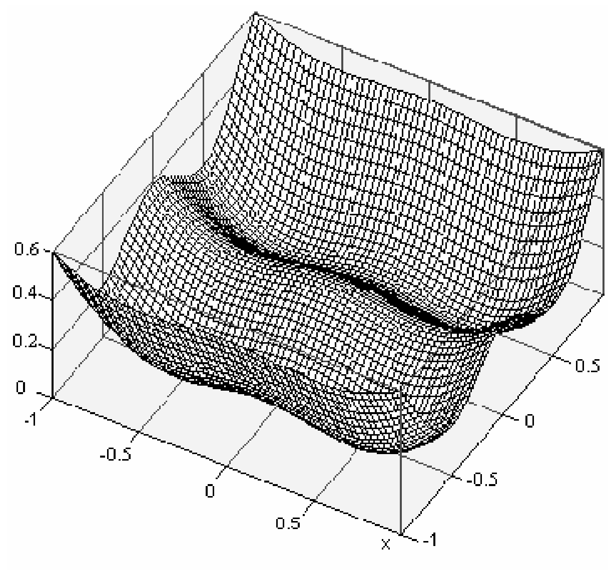

Fig. 9. The error reconstruction function for uniform sampling. The covariance function is Gaussian. The anisotropic case.

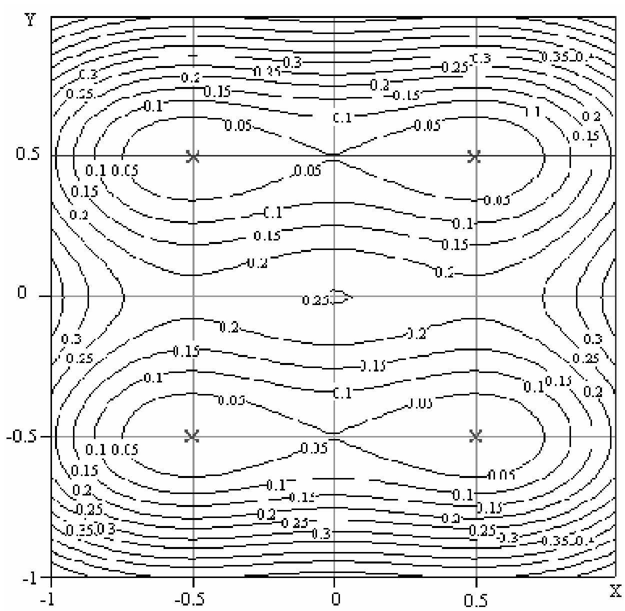

Fig. 10. The sections of the error reconstruction function for uniform sampling. The covariance function is Gaussian. Anisotropic case.

4 The non-uniform SRP

In this Section we consider three types of non-uniform sampling. There are polar, spiral, and arbitrary sampling.

4.1 Polar Sampling

In this type of sampling the sample points are located on concentric circles. Usually the distances between circles are the same. The location of the sample points on the circles is determined by the cross points of the circles and radii with different angles. Usually the angles between radii are identical. Below we used these rules of sampling.

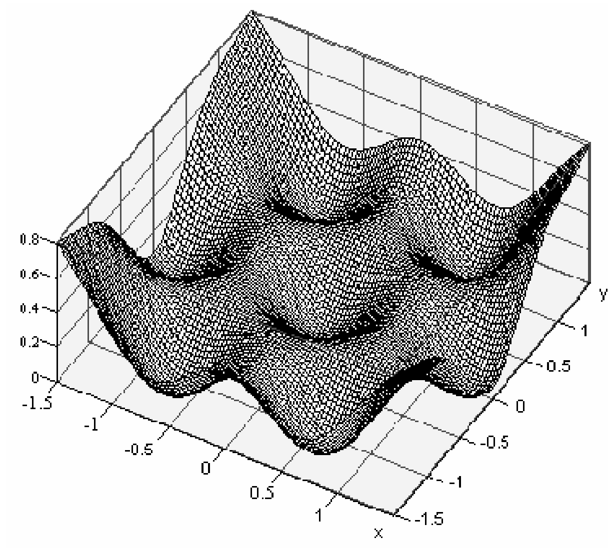

Fig. 11. The error reconstruction function for uniform sampling. The covariance function is Gaussian.

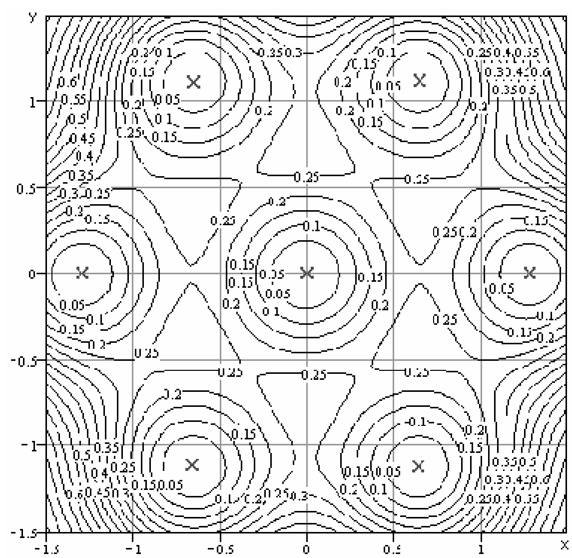

Fig. 12. The sections of the error reconstruction function for uniform sampling. The covariance function is Gaussian.

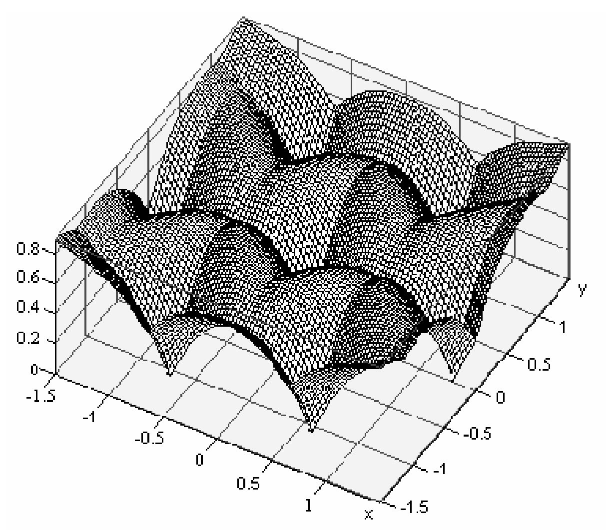

Fig. 13. The error reconstruction function for uniform sampling. The covariance function is exponential.

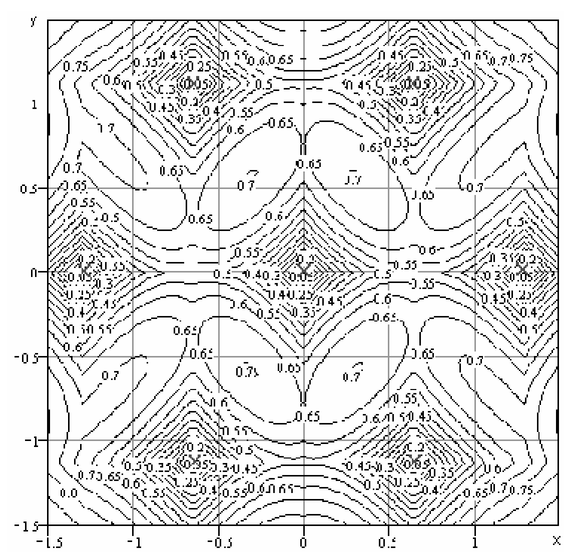

Fig. 14. The sections of error reconstruction function for uniform sampling. The covariance function is exponential.

The result of the calculations of the reconstruction function of the Gaussian field with the Gaussian covariance function and with the identical covariance radii $\rho_x = \rho_y = 0.67$ is presented in Fig. 15. The quantity of the radii is equal to 8, and the quantity of the rings is equal to 2. The distance Δr between the rings is equal to 1. The reconstruction error surfaces are presented in Fig. 16 and 17. Here the quantity of the radii is equal to 16 and the quantity of the rings is equal to 4. The distance Δr between the rings is equal to 1.2. The radii of the covariance are the same: $\rho_x = \rho_y = 0.67$. In Fig. 17 the points of samples are marked by crosses. One can see that the surface of the error has deep pits and high hills. The values of the error in the pits are equal to zero. The height of the hills is nearly equal to one. This effect is explained by the large distances between the rings ($\Delta r = 1.2$) in comparison with the radii of the covariance 0.67.

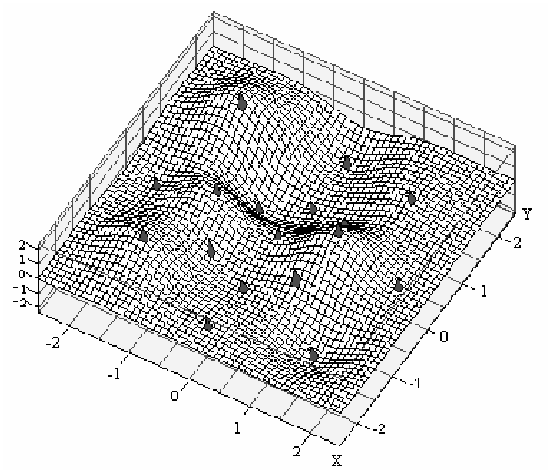

Fig. 15. The reconstruction function for polar sampling. The covariance function is Gaussian. The number of rings is 2, the number of radii is 8, the distance between the rings is 1, the covariance radii are equal to 0.67.

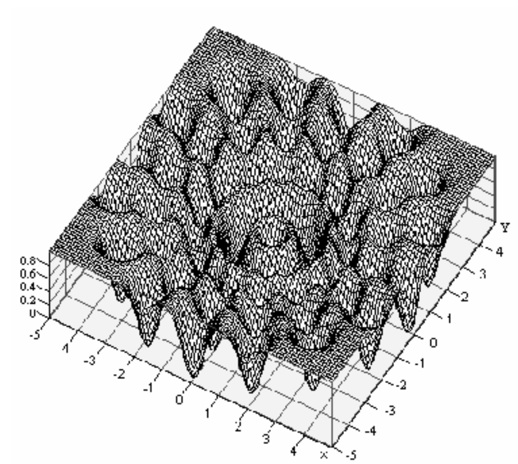

Fig. 16. The reconstruction function for polar sampling. The covariance function is Gaussian. The number of rings is 4, the number of radii is 16, the distance between the rings is 1.2, the covariance radii are equal to 0.67.

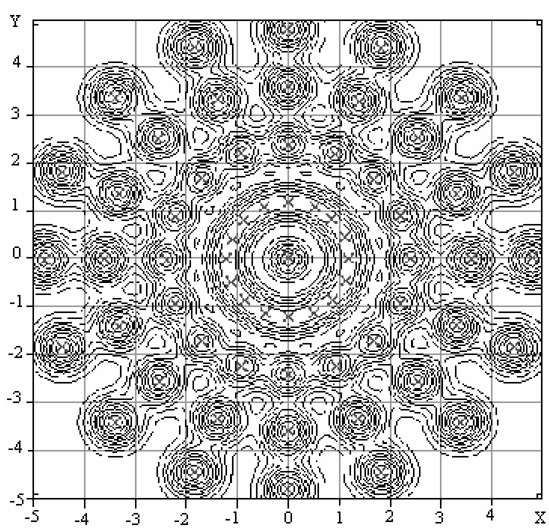

Fig. 17. The section of error reconstruction function for tpolar sampling. The covariance function is Gaussian.

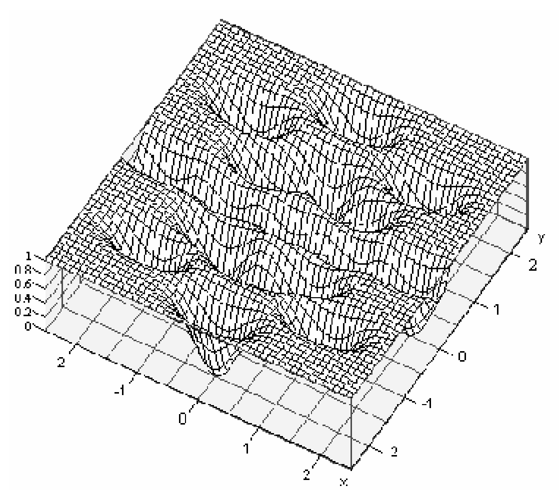

Fig. 18. The error reconstruction function for polar sampling. The covariance function is Gaussian. The anisotropic case.

The anisotropy case is illustrated by Fig. 18 and 19. Here there is the exponential covariance function. Once again, the distance ($\Delta r = 1$) between the rings is more then the radii of the covariance ($\rho_x = 0.67, \rho_y = 0.33$). So there are large hills on the surface of the error. It is natural that slopes are steeper along the *x* axis than along the *y* axis.

By the way, this algorithm gives a way to calculate the main characteristics of the polar SRP if the distances between rings are different and the angles between the radii are different as well.

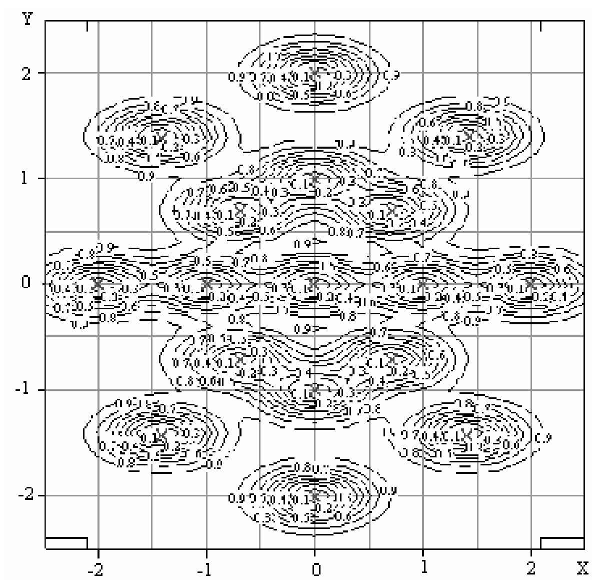

Fig. 19. The section of error reconstruction function for tpolar sampling. The covariance function is Gaussian. The anisotropic case.

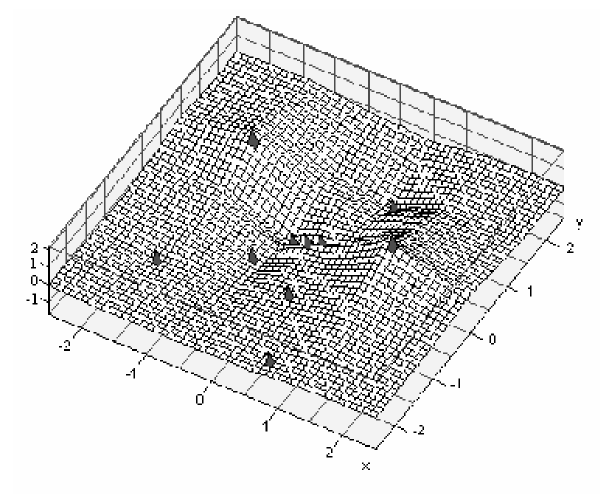

Fig. 20. The reconstruction function for spiral sampling. The covariance function is exponential.

4.2 Spiral Sampling

Linear spiral sampling is usually described by the following expression (see, Bourgeois, (2001)):

$$r_{jk} = \alpha(\varphi_k + 2\pi j), 0 \le \varphi_k < 2\pi \tag{13}$$

where r_{jk} is the sampling point on the revolution with the number *j* and on the row with the angle φ_k; α is a constant parameter of the spiral; $k = 0,1,2,..., N_\varphi - 1$; N_φ is the quantity of the rows crossing the spiral or the number of samples on any revolution; $j = 0,1,2,...$ is the number of the revolution. For our aim it is necessary to determine the Cartesian coordinates of each sampling point following (13). The method of the calculations is based on the formulas (11) and (12). One example of the surface of the reconstruction function of the field with the exponential covariance function is presented in Fig. 20. The sample points are marked by peaks. In Fig. 21 - 24 the surfaces of the reconstruction error function are given. The covariance function is Gaussian for both cases. In these figures all parameters are the same: the quantity of rings is equal to 2; the quantity of rows is equal to 5; the parameter $\alpha = 0.2$. There is one exception: in Fig 21 and 22 we have the isotropic case ($\rho_x = 1, \rho_y = 1$); in Fig. 23 and 24 there is the anisotropic case ($\rho_x = 1, \rho_y = 0.5$). This is the reason for the prolate character of the graphs in Fig. 23 and 24. In Fig. 21 - 24 one can see the central area with a very small error. It is very easy to understand this fact: the distances between samples in this area is rather small in comparison with the radii of the covariances.

There are not any difficulties in calculating the same SRP characteristics for any other types of sampling (a hyperbolic spiral, a logarithmic spiral, etc.).

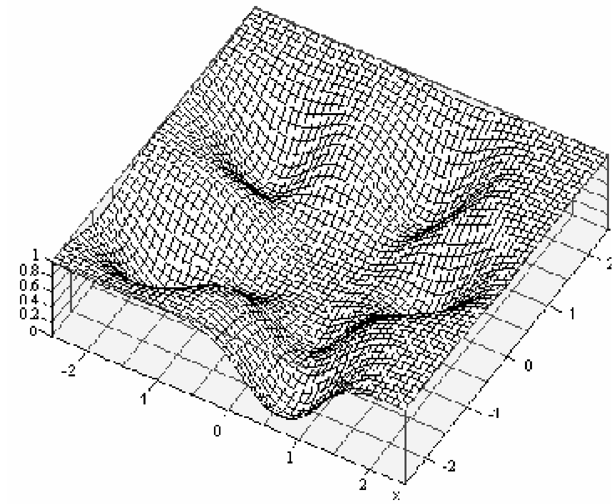

Fig. 21. The reconstruction function for spiral sampling. The covariance function is Gaussian.s

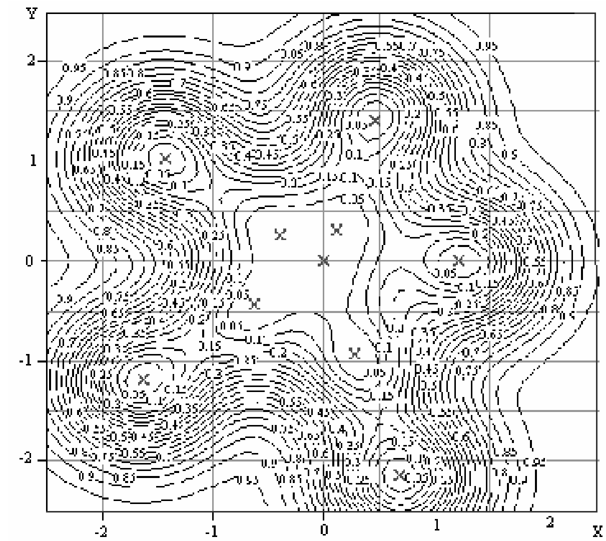

Fig. 22. The sections of the error reconstruction function for spiral sampling. The covariance function is Gaussian.

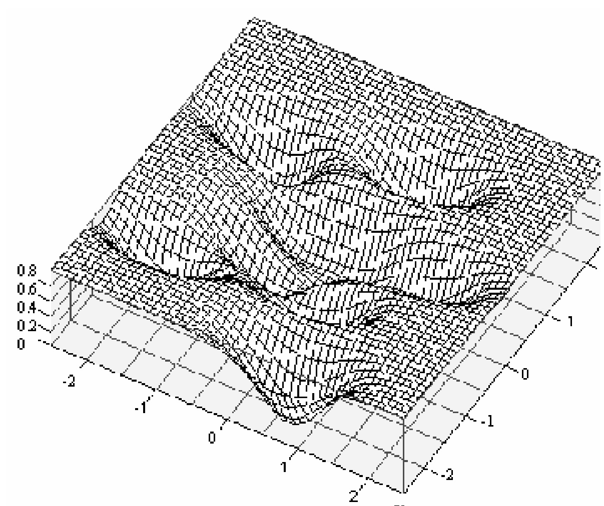

Fig. 23. The reconstruction function for spiral sampling. The covariance function is Gaussian. The anisotropic case

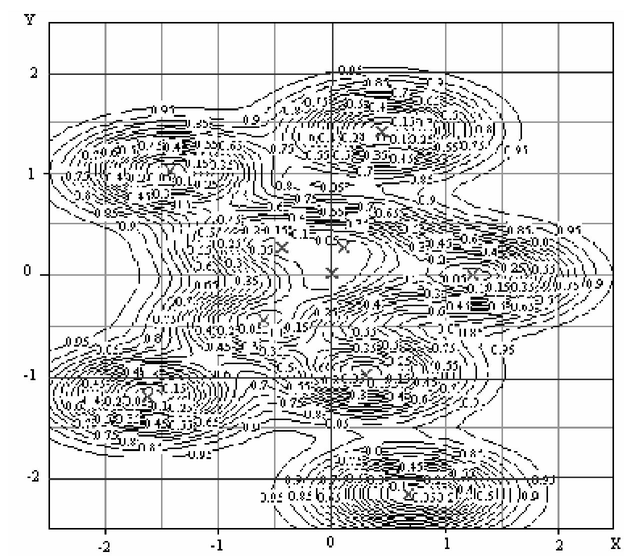

Fig. 24. The sections of the error reconstruction function for spiral sampling. The covariance function is Gaussian. The un-isotropic case.

4.3 Arbitrary Sampling

The statistical description of the SRP of the Gaussian fields on the basis of formulas (11) and (12) makes it possible to calculate the main SRP characteristics for an arbitrary location of samples. It is necessary to remember that there are not any problems with the SRP description of Gaussian stochastic processes with a non-uniform sampling. We can see the same situation in the statistical description of the Gaussian random fields. We illustrate this statement by the surfaces of the reconstruction error functions in Fig. 25 - 28. There are four samples on both figures. The covariance radii are equal: $\rho_x = \rho_y = 1$ in both cases. The Cartesian coordinates of the samples are following:

Fig. 25 and 26 $- x_1 = y_1 = -0.75$; $x_2 = -0.5; y_2 = 0.5$; $x_3 = 0.5; y_3 = 0.25$; $x_4 = 0.75; y_4 = -0.5$.

Fig. 27 and 28 $- x_1 = y_1 = -0.75$; $x_2 = -0.25; y_2 = 0.75$; $x_3 = -0.25; y_3 = 0$; $x_4 = 0.75; y_4 = -0.25$.

As one can see, the locations of the sample points are arbitrary. The type of the covariance function is Gaussian. The character of the graphs does not now demand any special comments.

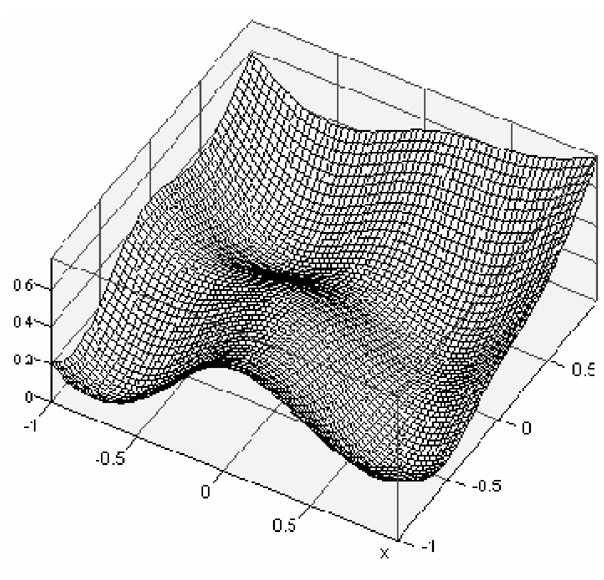

Fig. 25. The reconstruction function for arbitrary sampling. The covariance function is Gaussian.

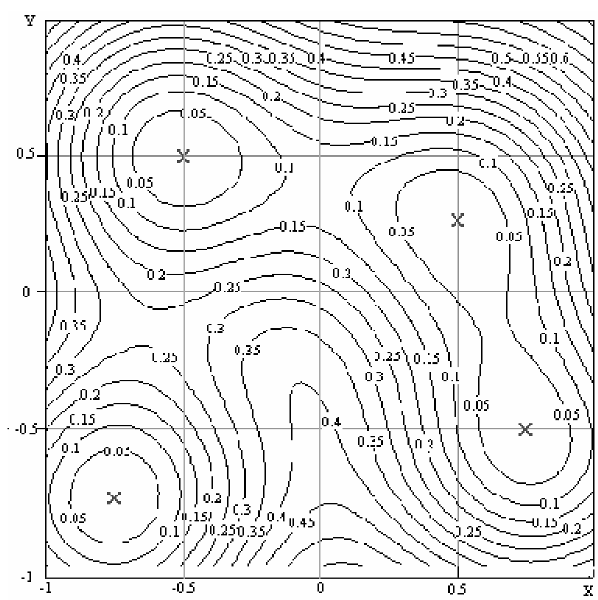

Fig. 26. The sections of error reconstruction function for arbitrary sampling. The covariance function is Gaussian.

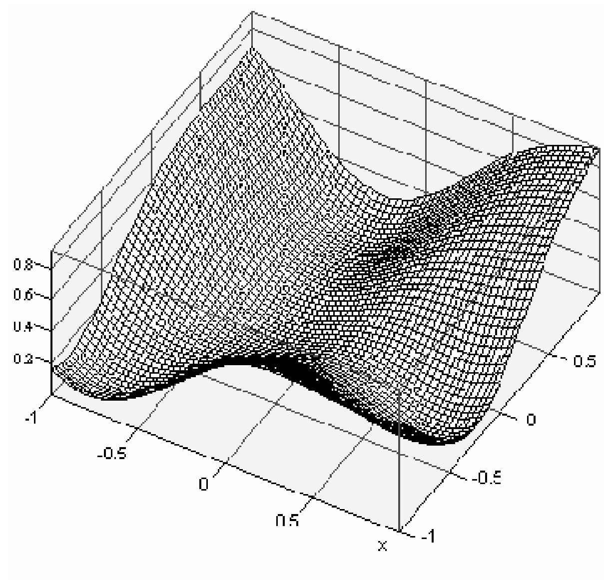

Fig. 27. The reconstruction function for arbitrary sampling. The covariance function is Gaussian.

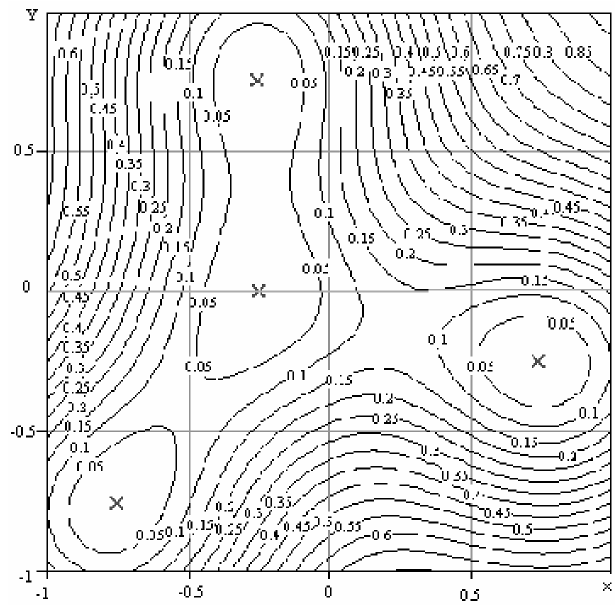

Fig. 28. The sections of error reconstruction function for arbitrary sampling. The covariance function is Gaussian.

5. Conclusions

The statistical description of the optimal Sampling-Reconstruction Procedure on the basis of the conditional mean rule was productively applied for many cases of stochastic processes. This work is the generalization of this approach for the optimal SRP description on random fields. We use the *usual* initial information in order to describe the random fields - their space covariance functions. But thanks to the fact that we concretize the type of probability density function of the fields (namely, the Gaussian pdf) we found it possible to apply the formulas for the conditional mean and for the conditional variance in order to describe the SRP of the Gaussian fields. We demonstrated the exceptional simplicity of the calculation of the main statistical characteristics of the SRP for many variants.

The Gaussian fields are described by two types of space covariance functions: exponential and Gaussian. The corresponding spectrums are infinite. There are not any restrictions on the view of the covariance function or for the type of power spectrums. For instance, the space spectrum can be limited. A lot of both reconstruction and error reconstruction surfaces are obtained by the numerical calculation. The applied algorithm of the statistical SRP description of Gaussian fields makes it possible to reflect a lot of SRP details. We demonstrate how both the reconstruction and the error reconstruction functions are changed if the covariance functions, the types of sampling (uniform: triangular, square, etc. and non-uniform: polar, spiral, and arbitrary), the quantity of samples, the distances between samples, and radii of the covariance functions of both axes are changed as well. The results of the calculations have clear physical interpretations.

Acknowledgement

This work has been partially supported by Consejo Nacional de Ciencia y Tecnologia (CONACYT) of Mexico under Project No 31472 and by National Polytecnical Institute of Mexico (IPN) under Project No 990350.

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